

# Existence of multiple solutions to a class of nonlinear Schrödinger system with external sources terms

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## Abstract

We study a class of nonlinear schrödinger system with external sources terms as perturbations in order to obtain existence of multiple solutions, this system arises from Bose-Einstein condensates etc.. As these external sources terms are positive functions and small in some sense, we use Nehari manifold to get the existence of a positive ground state solution and a positive bound state solution.

*Keywords:* nonlinear schrödinger system; multiple solutions; ground state; Nehari manifold

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# 1 Introduction

We are concerned with the following nonlinear schrödinger system with external sources terms

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta uv^2 + f(x), & x \in \Omega, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v + g(x), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ ,  $N \leq 3$ ,  $\lambda_1, \lambda_2, \mu_1, \mu_2$  and  $\beta$  are positive constants,  $f(x), g(x)$  are external sources terms.

(1.1) is a perturbed version of the following system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta uv^2, & x \in \Omega, \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \quad (1.2)$$

System (1.2) arises from many physical problems, especially in describing some phenomenon in nonlinear optics ([1],[8]). It also models the Hartee-Fock theory for a double condensate, i.e., a binary mixture of Bose-Einstein condensates in two different hyperfine states  $|1\rangle$  and  $|2\rangle$  ([15]). We refer the reader to [2, 3, 6, 7, 9, 10, 11, 12, 13, 17, 18, 19, 20, 23], and the references therein for interesting existence of solutions or properties of solutions. The parameters  $\mu_i$  and  $\beta$  are the intraspecies and interspecies scattering lengths respectively. The sign of the scattering length  $\beta$  determines whether the interaction of states  $|1\rangle$  and  $|2\rangle$  are repulsive or attractive. When  $\beta < 0$ , the interactions of states  $|1\rangle$  and  $|2\rangle$  are repulsive. In contrast, when  $\beta > 0$  the interactions of states  $|1\rangle$  and  $|2\rangle$  are attractive. For atoms of the single state  $|j\rangle$ , when  $\mu_j > 0$ , the interactions of the single state  $|j\rangle$  are attractive.

Naturally, people concern nontrivial solutions(solutions with both components nonzero) of the system. In recent years, many interesting works have been devoted to searching ground states and bound states for this system, see [2, 3, 6, 7, 17, 18, 23] etc. and the references therein.

**A positive ground state solution** we mean a solution of a schrödinger system which has the least energy among all nonzero solutions, and both of its components are positive. Note here we call a function positive if it is nonnegative and nonzero. A **bound state solution** refers to limited-energy solution. As for system (1.2), well-known results indicate the existence of positive ground state is closely related to the parameters, see [3],[23] etc. and a remark for the bounded-domain case in [16].

In this paper, multiplicity result is established when the perturbations are sufficiently small. If  $f(x)$  and  $g(x)$  are both positive, we can find a positive ground state.

To be precise, let  $S_4$  be the best Sobolev constant of the embedding:  $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ , then we have

**Theorem 1.1** *Assume that  $f(x), g(x) \in L^{\frac{4}{3}}(\Omega)$ , both nonzero. Then there exists a positive constant  $\Lambda = \Lambda(\lambda_1, \lambda_2, \mu_1, \mu_2, \beta, S_4)$ , such that whenever  $\max\{\|f\|_{\frac{4}{3}}, \|g\|_{\frac{4}{3}}\} < \Lambda$ , system (1.1) has two nontrivial solutions. Furthermore, if  $f$  and  $g$  are both positive, system (1.1) has one positive ground state solution and one positive bound state solution.*

Let  $H := H_0^1(\Omega) \times H_0^1(\Omega)$  with the norm

$$\|(u, v)\| := \left( \int_{\Omega} (|\nabla u|^2 + \lambda_1 u^2) + \int_{\Omega} (|\nabla v|^2 + \lambda_2 v^2) \right)^{\frac{1}{2}}.$$

An element  $(u, v) \in H$  is called a weak solution of (1.1), if the equality

$$\begin{aligned} \int_{\Omega} (\nabla u \nabla \varphi + \nabla v \nabla \psi - \lambda_1 u \varphi - \lambda_2 v \psi - \mu_1 u^3 \varphi - \mu_2 v^3 \psi \\ - \beta u v^2 \varphi - \beta u^2 v \psi - f \varphi - g \psi) dx = 0 \end{aligned}$$

holds for all  $(\varphi, \psi) \in H$ . A weak solution of (1.1) corresponds to a critical point of the following  $C^1$ -functional

$$J(u, v) = \frac{1}{2} \|(u, v)\|^2 - \frac{1}{4} (\mu_1 \|u\|_4^4 + \mu_2 \|v\|_4^4 + 2\beta \int_{\Omega} u^2 v^2) - \int_{\Omega} (f u + g v). \quad (1.3)$$

Denote the Nehari manifold associated with the functional by

$$\mathcal{N} := \{(u, v) \in H : \langle J'(u, v), (u, v) \rangle = 0\}.$$

It is well-known that all critical points lie in the Nehari manifold, and it is usually effective to consider the existence of critical points in this smaller subset of the Sobolev space. For fixed  $(u, v) \in H \setminus \{(0, 0)\}$ , denote

$$\phi(t) = \phi_{(u, v)}(t) := J(tu, tv), \quad t > 0 \quad (1.4)$$

the so called fibering map in the direction  $(u, v)$ . Such maps are often used to investigate Nehari manifolds for various semi-linear problems([4, 21, 22, 24]).

Our method is, roughly speaking, to figure out two non-degenerate parts of the Nehari manifold, and then consider minimization problems in the two parts respectively to obtain two nontrivial solutions, especially to obtain two positive solutions. Under some other assumptions, there may exist more solutions, after we submitted this manuscript, we thank the referee to tell us that existence of infinitely many nontrivial solutions of systems related to (1.1) are recently obtained in [25] by different methods under different conditions, they assume the coefficient matrix is either positive definite when  $N = 1, 2$  or anti-symmetric when  $N = 3$ . But here for (1.1), the coefficient matrix  $\begin{pmatrix} \mu_1 & \beta \\ \beta & \mu_2 \end{pmatrix}$  usually doesn't satisfy those assumptions, especially the anti-symmetric assumption implies that  $\beta = 0$  when  $N = 3$ , then (1.1) becomes two independent differential equations without any couplings. Our assumption here on  $\beta$  can be  $\beta > -\sqrt{\mu_1\mu_2}$  (notice the remark at the end of the paper).

Our paper is organized as follows. In section 2, we use fibering maps to divide the Nehari manifold into three parts. A basic work related can be found in [21],[22]. The number  $\Lambda$  in Theorem 1.1 is determined to ensure a satisfactory partition. In Section 3 we set up a critical point lemma to consider two minimization problems. In the last section, we give the proof of Theorem 1.1.

## 2 Partition of the Nehari manifold

Denote  $\Phi(u, v) := \langle J'(u, v), (u, v) \rangle$ . By (1.3) one has

$$\langle J'(u, v), (u, v) \rangle = \|(u, v)\|^2 - (\mu_1\|u\|_4^4 + \mu_2\|v\|_4^4 + 2\beta \int_{\Omega} u^2 v^2) - \int_{\Omega} (fu + gv),$$

$$\langle \Phi'(u, v), (u, v) \rangle = 2\|(u, v)\|^2 - 4(\mu_1\|u\|_4^4 + \mu_2\|v\|_4^4 + 2\beta \int_{\Omega} u^2 v^2) - \int_{\Omega} (fu + gv).$$

Now we divide the Nehari manifold into three parts:

$$\mathcal{N}^+ := \{(u, v) \in \mathcal{N} : \langle \Phi'(u, v), (u, v) \rangle > 0\};$$

$$\mathcal{N}^0 := \{(u, v) \in \mathcal{N} : \langle \Phi'(u, v), (u, v) \rangle = 0\};$$

$$\mathcal{N}^- := \{(u, v) \in \mathcal{N} : \langle \Phi'(u, v), (u, v) \rangle < 0\}.$$

Obviously, only  $\mathcal{N}^0$  contains the element  $(0, 0)$ , and it is easy to see  $\mathcal{N}^+ \cup \mathcal{N}^0$  and  $\mathcal{N}^- \cup \mathcal{N}^0$  are both closed subsets of  $H$ .

In order to make an explanation of such partition, for  $(u, v) \in H \setminus \{(0, 0)\}$ , let us consider the fibering map defined in (1.4). Since

$$\phi'(t) = \frac{1}{t} \langle J'(tu, tv), (tu, tv) \rangle,$$

we know  $(u, v) \in \mathcal{N}$  if and only if  $\phi'(1) = 0$ . Moreover,  $(tu, tv) \in \mathcal{N}$  with  $t > 0$  if and only if  $\phi'(t) = 0$ .

Thus for a fixed direction  $(u, v) \in H \setminus \{(0, 0)\}$ , we can obtain all elements on the Nehari manifold which lie in this direction if we can find all stationary points of the fibering map. As a result, one would obtain all nonzero elements of the Nehari manifold if one could find stationary points of fibering maps in all directions. We remark that the number of roots of the equation  $\phi'(t) = 0$  does not depend on the norm of  $(u, v)$ , once this direction is fixed. Indeed, for  $\delta > 0$  we have

$$\phi'_{(\delta u, \delta v)}\left(\frac{t}{\delta}\right) = \delta \phi'_{(u, v)}(t).$$

Thus

$$\phi'_{(u, v)}(t) = 0, \ t > 0 \iff \phi'_{(\delta u, \delta v)}\left(\frac{t}{\delta}\right) = 0, \ t > 0.$$

Furthermore,

$$\phi''_{(\delta u, \delta v)}\left(\frac{t}{\delta}\right) = \delta^2 \phi''_{(u, v)}(t),$$

which means it is also irrelevant with the norm of  $(u, v)$  if we consider the sign of second derivative of the fibering map at its stationary points. Moreover, stationary points can be classified into three types, namely local minimum, local maximum and turning point, according to the sign of second derivative of the fibering map at these points.

We now disclose the relationship between such classification and the former partition of the Nehari manifold. By direct calculation, we get

$$\phi'(t) = \frac{1}{t} \langle J'(tu, tv), (tu, tv) \rangle = \frac{1}{t} \Phi(tu, tv),$$

$$\phi''(t) = \frac{1}{t^2}[\langle \Phi'(tu, tv), (tu, tv) \rangle - \Phi(tu, tv)].$$

Thus if  $\phi'(t) = 0$ , then  $(tu, tv) \in \mathcal{N}$ , and  $\Phi(tu, tv) = 0$ , which yields

$$\phi''(t) = \frac{1}{t^2} \langle \Phi'(tu, tv), (tu, tv) \rangle.$$

Now it is easy to check:

$$t(u, v) \in \mathcal{N}^+, \quad t > 0 \iff \phi'(t) = 0, \phi''(t) > 0;$$

$$t(u, v) \in \mathcal{N}^0, \quad t > 0 \iff \phi'(t) = 0, \phi''(t) = 0;$$

$$t(u, v) \in \mathcal{N}^-, \quad t > 0 \iff \phi'(t) = 0, \phi''(t) < 0.$$

We can explore the Nehari manifold through fibering maps. In fact, we give the following important lemma to show when the degenerate part of the Nehari manifold is clear and simple.

To simplify the calculation, let us introduce some notations that will be used repeatedly in the rest. For  $(u, v) \in H$ , define

$$A = A(u, v) := \mu_1 \|u\|_4^4 + \mu_2 \|v\|_4^4 + 2\beta \int_{\Omega} u^2 v^2, \quad (2.1)$$

$$B = B(u, v) := \int_{\Omega} (fu + gv). \quad (2.2)$$

**Lemma 2.1** *Suppose that  $f(x), g(x) \in L^{\frac{4}{3}}(\Omega)$ , both nonzero, then there exists a positive constant  $\Lambda = \Lambda(\lambda_1, \lambda_2, \mu_1, \mu_2, \beta, S_4)$  such that  $\mathcal{N}^0 = \{(0, 0)\}$  when  $\max\{\|f\|_{\frac{4}{3}}, \|g\|_{\frac{4}{3}}\} < \Lambda$ .*

**Proof.** By the analysis above, we only need to prove for each  $(u, v) \in H$  with  $\|(u, v)\| = 1$ , the fibering map  $\phi(t) = \phi_{(u,v)}(t)$  has no stationary point that is a turning point. By the notations (2.1) and (2.2), we can write

$$\begin{aligned} \phi(t) &= J(tu, tv) \\ &= \frac{t^2}{2} \|(u, v)\|^2 - \frac{t^4}{4} (\mu_1 \|u\|_4^4 + \mu_2 \|v\|_4^4 + 2\beta \int_{\Omega} u^2 v^2) - t \int_{\Omega} (fu + gv) \\ &= \frac{t^2}{2} - \frac{At^4}{4} - Bt. \end{aligned}$$

Thus

$$\phi'(t) = t - At^3 - B, \quad \phi''(t) = 1 - 3At^2.$$

Notice  $A = A(u, v) = \mu_1 \|u\|_4^4 + \mu_2 \|v\|_4^4 + 2\beta \int_{\Omega} u^2 v^2 > 0$ . Define  $\psi(t) = t - At^3$ ,  $t > 0$ , then  $\phi'(t) = 0 \iff \psi(t) = B$ .

Let us consider the graph of  $\psi(t)$ :  $\psi''(t) = -6At < 0$ , so  $\psi(t)$  is strictly concave;  $\psi'(t) = 0 \iff t = \sqrt{\frac{1}{3A}}$ , thus  $\psi(t)$  takes its maximum at  $t = \sqrt{\frac{1}{3A}}$ , the value of which is  $\frac{2}{3}\sqrt{\frac{1}{3A}}$ . Also,  $\lim_{t \rightarrow 0^+} \psi(t) = 0$ ,  $\psi(\infty) = -\infty$ . All the above implies if  $0 < B < \frac{2}{3}\sqrt{\frac{1}{3A}}$ , the equation  $\phi'(t) = 0$  has exactly two roots  $t_1, t_2$  satisfying  $0 < t_1 < \sqrt{\frac{1}{3A}} < t_2$ ; if  $B \leq 0$  the equation  $\phi'(t) = 0$  has exactly one root denoted  $t'_2$ , which satisfies  $\sqrt{\frac{1}{3A}} < t'_2$ . Since  $\phi''(t) = 1 - 3At^2$ , considering the above two cases we have  $\phi''(t_1) > 0$ ,  $\phi''(t_2) < 0$ , and  $\phi''(t'_2) < 0$ . So when  $0 < B < \frac{2}{3}\sqrt{\frac{1}{3A}}$ , one has  $t_1(u, v) \in \mathcal{N}^+$ ,  $t_2(u, v) \in \mathcal{N}^-$ ; when  $B \leq 0$  one has  $t'_2(u, v) \in \mathcal{N}^-$ . Since  $f(x), g(x)$  are both nonzero, it is easy to check that the sets  $\{(u, v) \in H : \|(u, v)\| = 1, 0 < B < \frac{2}{3}\sqrt{\frac{1}{3A}}\} \neq \emptyset$  and  $\{(u, v) \in H : \|(u, v)\| = 1, B \leq 0\} \neq \emptyset$ , so  $\mathcal{N}^+ \neq \emptyset$  and  $\mathcal{N}^- \neq \emptyset$ . To finish the proof, we are in a position to determine a number  $\Lambda$  such that whenever  $\max\{\|f\|_{\frac{4}{3}}, \|g\|_{\frac{4}{3}}\} < \Lambda$  we have

$$B < \frac{2}{3}\sqrt{\frac{1}{3A}}. \quad (2.3)$$

In fact, since  $(u, v)$  lies on the unit sphere of  $H$ , by Sobolev inequality and Hölder inequality one obtains an upper bound for  $A$ , which yields the existence of a positive constant  $\alpha = \alpha(\lambda_1, \lambda_2, \mu_1, \mu_2, \beta, S_4)$  such that

$$0 < \alpha \leq \frac{2}{3}\sqrt{\frac{1}{3 \sup_{\|(u,v)\|=1} A(u,v)}}. \quad (2.4)$$

On the other hand, by Sobolev inequality and Hölder inequality, we obtain that

$$B = \int_{\Omega} (fu + gv) \leq \|f\|_{\frac{4}{3}} \|u\|_4 + \|g\|_{\frac{4}{3}} \|v\|_4 \leq \sqrt{2} S_4 \max\{\|f\|_{\frac{4}{3}}, \|g\|_{\frac{4}{3}}\} \|(u, v)\|.$$

That is,

$$B \leq \sqrt{2} S_4 \max\{\|f\|_{\frac{4}{3}}, \|g\|_{\frac{4}{3}}\}. \quad (2.5)$$

Now take  $\Lambda := \Lambda(\lambda_1, \lambda_2, \mu_1, \mu_2, \beta, S_4) = \frac{\alpha}{\sqrt{2}S_4}$ , by (2.4),(2.5) we know when  $\max\{\|f\|_{\frac{4}{3}}, \|g\|_{\frac{4}{3}}\} < \Lambda$ , (2.3) holds.  $\square$

### 3 Reduction to minimization problems

Here we reduce our discussion into two minimization problems through a critical point lemma. As to constraint minimization problems, we refer the reader to [5]. Let  $X, Y$  be real Banach spaces,  $U \subset X$  be an open set. Suppose that  $f : U \rightarrow \mathbb{R}^1$ ,  $g : U \rightarrow Y$  are  $C^1$  mappings in this section. Let

$$M = \{x \in U : g(x) = \theta\}.$$

For the following minimization problem

$$\min_{x \in M} f(x), \tag{3.1}$$

we have

**Lemma 3.1** ([5], Theorem 4.1.1) *Suppose that  $x_0 \in M$  solves (3.1), and that  $\text{Im } g'(x_0)$  is closed. Then there exists  $(\lambda, y^*) \in \mathbb{R}^1 \times Y^*$  such that  $(\lambda, y^*) \neq (0, \theta)$ , and*

$$\lambda f'(x_0) + g'(x_0)^* y^* = 0.$$

*Furthermore, if  $\text{Im } g'(x_0) = Y$ , then  $\lambda \neq 0$ .*

Let us introduce two minimization problems

$$\theta^+ = \inf_{(u,v) \in \mathcal{N}^+} J(u, v); \tag{3.2}$$

$$\theta^- = \inf_{(u,v) \in \mathcal{N}^-} J(u, v). \tag{3.3}$$

**Lemma 3.2** *If  $(u_1, v_1)$  solves (3.2), then  $(u_1, v_1)$  is a nontrivial weak solution of (1.1); if  $(u_2, v_2)$  solves (3.3), then  $(u_2, v_2)$  is a nontrivial weak solution of (1.1).*

**Proof.** We prove the first assertion, since the second is similar.

Recall  $\Phi(u, v) = \langle J'(u, v), (u, v) \rangle$  and



$$\mathcal{N}^+ = \{(u, v) \in \mathcal{N} : \langle \Phi'(u, v), (u, v) \rangle > 0\}.$$

Let  $U = \{(u, v) \in H : \langle \Phi'(u, v), (u, v) \rangle > 0\}$ , and rewrite

$$\mathcal{N}^+ = \{(u, v) \in U : \Phi(u, v) = 0\}.$$

Now we use Lemma 3.1 to consider problem (3.2) by taking  $X = H$ ,  $Y = \mathbb{R}^1$ ,  $U = \{(u, v) \in H : \langle \Phi'(u, v), (u, v) \rangle > 0\}$ ,  $f = J$ ,  $g = \Phi$ ,  $M = \mathcal{N}^+$ . Then there exists a real pair  $(\lambda_1, \lambda_2) \neq (0, 0)$ , satisfying

$$\lambda_1 J'(u_1, v_1) = \lambda_2 \Phi'(u_1, v_1).$$

Since  $\langle \Phi'(u_1, v_1), (u_1, v_1) \rangle > 0$ , we have  $\Phi'(u_1, v_1) \neq 0$ , again by Lemma 3.1, one may assume  $\lambda_1 \neq 0$ . In other words, there exists a real number  $\mu$  such that

$$J'(u_1, v_1) = \mu \Phi'(u_1, v_1). \quad (3.4)$$

Since

$$\mu \langle \Phi'(u_1, v_1), (u_1, v_1) \rangle = \langle J'(u_1, v_1), (u_1, v_1) \rangle = 0$$

and  $\langle \Phi'(u_1, v_1), (u_1, v_1) \rangle > 0$ , we obtain  $\mu = 0$ . By (3.4) we get  $J'(u_1, v_1) = 0$ . Obviously  $(u_1, v_1) \neq (0, 0)$ , noticing that (1.1) doesn't admit semi-trivial (nonzero but one component being trivial) solutions, we know that  $(u_1, v_1)$  is a nontrivial weak solution of (1.1).  $\square$

## 4 Proof of the main result

When  $\mathcal{N}^0 = \{(0, 0)\}$ , we show the solvability of the two problems (3.2) and (3.3), and give the proof of Theorem 1.1. The following lemma implies that  $\theta^+$  and  $\theta^-$  are both finite, and any minimizing sequence for (3.2) or (3.3) are bounded.

**Lemma 4.1** *Assume that  $f(x), g(x) \in L^{\frac{4}{3}}(\Omega)$ , then  $J$  is coercive and bounded from below on  $\mathcal{N}$  (thus on  $\mathcal{N}^+$  and  $\mathcal{N}^-$ ).*

**Proof.** Let  $(u, v) \in \mathcal{N}$ , from the definition of the Nehari manifold we have

$$\|(u, v)\|^2 - \int_{\Omega} (fu + gv) = (\mu_1 \|u\|_4^4 + \mu_2 \|v\|_4^4 + 2\beta \int_{\Omega} u^2 v^2),$$

this equality together with (1.3) yield

$$J(u, v) = \frac{1}{4} \|(u, v)\|^2 - \frac{3}{4} \int_{\Omega} (fu + gv). \quad (4.1)$$

Note that

$$\int_{\Omega} (fu + gv) \leq \|f\|_{\frac{4}{3}} \|u\|_4 + \|g\|_{\frac{4}{3}} \|v\|_4 \leq \sqrt{2} S_4 \max\{\|f\|_{\frac{4}{3}}, \|g\|_{\frac{4}{3}}\} \|(u, v)\|, \quad (4.2)$$

combining (4.1) and (4.2) we obtain

$$J(u, v) \geq \frac{1}{4} \|(u, v)\|^2 - \frac{3\sqrt{2}}{4} S_4 \max\{\|f\|_{\frac{4}{3}}, \|g\|_{\frac{4}{3}}\} \|(u, v)\|. \quad (4.3)$$

Since the right hand side of the inequality (4.3) is a quadratic function of  $\|(u, v)\|$ , it is easy to know that  $J$  is coercive and bounded from below on  $\mathcal{N}$ .  $\square$

From now on,  $\Lambda$  refers to the number in Lemma 2.1. Although the lemma below is valid under weaker conditions, we would assume  $f(x), g(x) \in L^{\frac{4}{3}}(\Omega)$ , both nonzero, and  $\max\{\|f\|_{\frac{4}{3}}, \|g\|_{\frac{4}{3}}\} < \Lambda$  in order to ensure  $\mathcal{N}^0 = \{(0, 0)\}$  by Lemma 2.1, because our discussion bases on a good partition of the Nehari manifold. We will often use the proof of Lemma 2.1 to analyze some properties of the manifold as well as associated fibering maps.

For the first minimization problem, we establish:

**Lemma 4.2** *Assume that  $f(x), g(x) \in L^{\frac{4}{3}}(\Omega)$ , both nonzero, and that  $\max\{\|f\|_{\frac{4}{3}}, \|g\|_{\frac{4}{3}}\} < \Lambda$ . Then  $\theta^+ < 0$ .*

**Proof.** By the proof of Lemma 2.1, we may take one  $(u, v) \in H$ ,  $\|(u, v)\| = 1$  such that  $0 < B < \frac{2}{3} \sqrt{\frac{1}{3A}}$ . Now the derivative  $\phi'(t)$  of the fibering map in this direction has exactly two positive zeros:  $t_1, t_2$ , satisfying  $0 < t_1 < \sqrt{\frac{1}{3A}} < t_2$ , and  $t_1 \cdot (u, v) \in \mathcal{N}^+$ . By  $\phi'(t) = t - At^3 - B$  we have  $\lim_{t \rightarrow 0^+} \phi'(t) = -B < 0$ , and  $\phi''(t) > 0, \forall t \in (0, \sqrt{\frac{1}{3A}})$ . Since  $\phi'(t_1) = 0$ , we know that  $\phi(t_1) < \lim_{t \rightarrow 0^+} \phi(t) = 0$ . Since  $\phi(t_1) = J(t_1 u, t_1 v) \geq \theta^+$ , we obtain that  $\theta^+ < 0$ .  $\square$

For the second minimization problem we need the following lemma, from which we know  $\mathcal{N}^-$  stays away from the origin.

**Lemma 4.3** *Assume that  $f(x), g(x) \in L^{\frac{4}{3}}(\Omega)$ , both nonzero, and that  $\max\{\|f\|_{\frac{4}{3}}, \|g\|_{\frac{4}{3}}\} < \Lambda$ . Then  $\mathcal{N}^-$  is closed.*

**Proof.** Under the assumptions one has

$$cl(\mathcal{N}^-) \subset \mathcal{N}^- \cup \mathcal{N}^0 = \mathcal{N}^- \cup \{(0, 0)\},$$

where we denote  $cl(\mathcal{N}^-)$  the closure of  $\mathcal{N}^-$ . Thus we only need to prove  $(0, 0) \notin cl(\mathcal{N}^-)$ , which is equivalent to prove that

$$dist((0, 0), \mathcal{N}^-) > 0.$$

Take  $(u, v) \in \mathcal{N}^-$ , and denote

$$(\tilde{u}, \tilde{v}) = \frac{(u, v)}{\|(u, v)\|},$$

then  $\|(\tilde{u}, \tilde{v})\| = 1$ . Under the assumptions, by the proof of Lemma 2.1, we obtain that  $A := A(\tilde{u}, \tilde{v})$ ,  $B := B(\tilde{u}, \tilde{v})$  satisfy  $B < \frac{2}{3}\sqrt{\frac{1}{3A}}$ . Furthermore, if  $0 < B < \frac{2}{3}\sqrt{\frac{1}{3A}}$ , the equation  $\phi'_{(\tilde{u}, \tilde{v})}(t) = 0$  has exactly two roots also denoted by  $t_1, t_2$ , which satisfy  $t_2 \cdot (\tilde{u}, \tilde{v}) \in \mathcal{N}^-$ ,  $t_1 \cdot (\tilde{u}, \tilde{v}) \in \mathcal{N}^+$ , we have  $t_2 \cdot (\tilde{u}, \tilde{v}) = (u, v)$ , so  $t_2 = \|(u, v)\|$ . If  $B \leq 0$ , then the equation  $\phi'_{(\tilde{u}, \tilde{v})}(t) = 0$  has exactly one root still denoted by  $\tilde{t}_2$ , thus we get  $\tilde{t}_2 \cdot (\tilde{u}, \tilde{v}) = (u, v)$ , then  $\tilde{t}_2 = \|(u, v)\|$ . Since  $t_2 > \sqrt{\frac{1}{3A}}$ ,  $\tilde{t}_2 > \sqrt{\frac{1}{3A}}$  in the proof of Lemma 2.1, so no matter which of the above two cases happens, we always obtain  $\|(u, v)\| > \sqrt{\frac{1}{3A}}$ . Noticing that  $A$  is bounded from above, we know that there exists  $\tau > 0$  such that

$$\|(u, v)\| > \tau.$$

We obtain

$$dist((0, 0), \mathcal{N}^-) = \inf_{(u, v) \in \mathcal{N}^-} \|(u, v)\| \geq \tau > 0,$$

which completes the proof.  $\square$

In order to abstract a  $(PS)_{\theta+}$  sequence from the minimizing sequence for problem (3.2), we use the idea of [22] to obtain the following lemma.

**Lemma 4.4** Assume that  $f(x), g(x) \in L^{\frac{4}{3}}(\Omega)$ , both nonzero. Then for  $(u, v) \in \mathcal{N}^+$ , there exists  $\epsilon = \epsilon(u, v) > 0$  and a differentiable function  $\xi^+ : B_\epsilon(0, 0) \rightarrow \mathbb{R}_+ := (0, +\infty)$  such that

- $\xi^+(0, 0) = 1$ ;
- $\xi^+(w, z)(u - w, v - z) \in \mathcal{N}^+, \forall (w, z) \in B_\epsilon(0, 0)$ ;
- $\langle (\xi^+)'(0, 0), (w, z) \rangle = [|| (u, v) ||^2 - 3A(u, v)]^{-1} [2 \int_{\Omega} (\nabla u \nabla w + \lambda_1 u w + \nabla v \nabla z + \lambda_2 v z) - 4 \int_{\Omega} (\mu_1 u^3 w + \mu_2 v^3 z + \beta u v^2 w + \beta u^2 v z) - \int_{\Omega} (f w + g z)]$ .

**Proof.** Define a  $C^1$ -mapping  $F : \mathbb{R}_+ \times H \rightarrow \mathbb{R}$  as follows:

$$F(t, (w, z)) = t ||(u - w, v - z)||^2 - t^3 A(u - w, v - z) - \int_{\Omega} [f \cdot (u - w) + g \cdot (v - z)].$$

By (2.1), we know that  $A(u - w, v - z) = \mu_1 ||u - w||_4^4 + \mu_2 ||v - z||_4^4 + 2\beta \int_{\Omega} (u - w)^2 (v - z)^2$ . Since  $(u, v) \in \mathcal{N}^+$  we have

$$F(1, (0, 0)) = 0.$$

Consider the fibering map  $\phi(t) = \phi_{(u, v)}(t) := J(tu, tv)$ , since

$$F(t, (0, 0)) = t ||(u, v)||^2 - t^3 (\mu_1 ||u||_4^4 + \mu_2 ||v||_4^4 + 2\beta \int_{\Omega} u^2 v^2) - \int_{\Omega} (fu + gv),$$

we have

$$F(t, (0, 0)) = \phi'(t).$$

Since  $(u, v) \in \mathcal{N}^+$ , we get  $\phi''(1) > 0$ , thus

$$\frac{\partial F}{\partial t}(1, (0, 0)) = \phi''(1) > 0.$$

We apply the implicit function theorem at point  $(1, (0, 0))$  to obtain the existence of  $\epsilon = \epsilon(u, v) > 0$  and differentiable function  $\xi^+(u, v)$  (i.e.,  $t(u, v)$ ) :  $B_\epsilon(0, 0) \rightarrow \mathbb{R}_+$  such that

- $\xi^+(0, 0) = 1$ ;

- $\xi^+(w, z) \cdot (u - w, v - z) \in \mathcal{N}, \forall (w, z) \in B_\epsilon(0, 0).$

Besides, we obtain the third conclusion of this lemma by calculation. To finish the second one, we only need to choose  $\epsilon = \epsilon(u, v) > 0$  small, such that  $\xi^+(w, z)(u - w, v - z) \in \mathcal{N}^+, \forall (w, z) \in B_\epsilon(0, 0)$ . Indeed, since  $\mathcal{N}^- \cup \mathcal{N}^0$  is closed,  $\text{dist}((u, v), \mathcal{N}^- \cup \mathcal{N}^0) > 0$ . Since  $\xi^+(w, z) \cdot (u - w, v - z)$  is continuous with respect to  $(w, z)$ , when  $\epsilon = \epsilon(u, v) > 0$  is small enough, we know

$$\|\xi^+(w, z) \cdot (u - w, v - z) - (u, v)\| < \frac{1}{2} \text{dist}((u, v), \mathcal{N}^- \cup \mathcal{N}^0), \forall (w, z) \in B_\epsilon(0, 0).$$

That is,  $\xi^+(w, z) \cdot (u - w, v - z)$  does not belong to  $\mathcal{N}^- \cup \mathcal{N}^0$ . Thus  $\xi^+(w, z) \cdot (u - w, v - z) \in \mathcal{N}^+$  and our proof is completed.  $\square$

Similarly, we can establish the following lemma, which will be used to abstract a  $(PS)_{\theta^-}$  sequence from the minimizing sequence for problem (3.3).

**Lemma 4.5** *Assume that  $f(x), g(x) \in L^{\frac{4}{3}}(\Omega)$ , both nonzero. Then for  $(u, v) \in \mathcal{N}^-$ , there exists  $\epsilon = \epsilon(u, v) > 0$  and a differentiable function  $\xi^- : B_\epsilon(0, 0) \rightarrow \mathbb{R}_+$  such that*

- $\xi^-(0, 0) = 1;$
- $\xi^-(w, z)(u - w, v - z) \in \mathcal{N}^-, \forall (w, z) \in B_\epsilon(0, 0);$
- $\langle (\xi^-)'(0, 0), (w, z) \rangle = [\|(u, v)\|^2 - 3A(u, v)]^{-1} [2 \int_{\Omega} (\nabla u \nabla w + \lambda_1 u w + \nabla v \nabla z + \lambda_2 v z) - 4 \int_{\Omega} (\mu_1 u^3 w + \mu_2 v^3 z + \beta u v^2 w + \beta u^2 v z) - \int_{\Omega} (f w + g z)].$

We are in a position to give:

**Lemma 4.6** *Assume that  $f(x), g(x) \in L^{\frac{4}{3}}(\Omega)$ , both nonzero, and that  $\max\{\|f\|_{\frac{4}{3}}, \|g\|_{\frac{4}{3}}\} < \Lambda$ . Then there exists a sequence  $\{(u_n, v_n)\} \subset \mathcal{N}^+$  such that  $(n \rightarrow \infty)$ :*

1.  $J(u_n, v_n) \rightarrow \theta^+;$
2.  $J'(u_n, v_n) \rightarrow 0.$

**Proof.** Notice  $\mathcal{N}^+ \cup \{(0, 0)\}$  is closed in  $H$ , we use the Ekeland's variational principle([14]) on  $\mathcal{N}^+ \cup \{(0, 0)\}$  to obtain a minimizing sequence  $\{(u_n, v_n)\} \subset \mathcal{N}^+ \cup \{(0, 0)\}$  such that

$$(a) \quad J(u_n, v_n) < \inf_{(u,v) \in \mathcal{N}^+ \cup \{(0,0)\}} J(u, v) + \frac{1}{n};$$

$$(b) \quad J(w, z) \geq J(u_n, v_n) - \frac{1}{n} \|(w - u_n, z - v_n)\|, \quad \forall (w, z) \in \mathcal{N}^+ \cup \{(0,0)\}.$$

By Lemma 4.2, we know  $\theta^+ < 0$ ; since  $J(0,0) = 0$ , we get that

$$\inf_{(u,v) \in \mathcal{N}^+ \cup \{(0,0)\}} J(u, v) = \theta^+.$$

Thus  $J(u_n, v_n) \rightarrow \theta^+$ , we may assume  $\{(u_n, v_n)\} \subset \mathcal{N}^+$ , then the first assertion holds.

For the second assertion, Firstly we have that  $\inf_n \|(u_n, v_n)\| \geq m > 0$ , where  $m$  is a constant. Indeed, if not, then  $J(u_n, v_n)$  would converge to zero. Moreover, by Lemma 4.1 we know that  $J$  is coercive on  $\mathcal{N}^+$ , then  $\{\|(u_n, v_n)\|\}$  is bounded. i.e.,  $\exists M > 0$  such that

$$0 < m \leq \|(u_n, v_n)\| \leq M. \quad (4.4)$$

Now by contradiction we assume  $\|J'(u_n, v_n)\| \geq C > 0$  as  $n$  is large, otherwise we may extract a subsequence to get the conclusion.

Now let us take  $(u, v) = (u_n, v_n)$  in Lemma 4.4, and define a differentiable function  $\xi_n^+ : (-\epsilon, \epsilon) \rightarrow \mathbb{R}_+$  as:

$$\xi_n^+(\delta) := \xi^+ \left( \frac{\delta J'(u_n, v_n)}{\|J'(u_n, v_n)\|} \right),$$

then by Lemma 4.4, we know that  $\xi_n^+(0) = \xi^+(0,0) = 1$ , and for  $\delta \in (-\epsilon, \epsilon)$  we have

$$(w, z)_\delta := \xi_n^+(\delta) \cdot [(u_n, v_n) - \frac{\delta J'(u_n, v_n)}{\|J'(u_n, v_n)\|}] \in \mathcal{N}^+.$$

Since  $(u_n, v_n)$  satisfies (b), one has

$$J(u_n, v_n) - J((w, z)_\delta) \leq \frac{1}{n} \|(w, z)_\delta - (u_n, v_n)\|. \quad (4.5)$$

Expanding the left hand side of (4.5) we get

$$J(u_n, v_n) - J((w, z)_\delta) = (1 - \xi_n^+(\delta)) \langle J'((w, z)_\delta), (u_n, v_n) \rangle$$

$$\begin{aligned}
& + \delta \xi_n^+(\delta) \langle J'((w, z)_\delta), \frac{J'(u_n, v_n)}{\|J'(u_n, v_n)\|} \rangle \\
& + o(\|(w, z)_\delta - (u_n, v_n)\|). \tag{4.6}
\end{aligned}$$

Combining (4.6) with (4.5) we obtain

$$\begin{aligned}
& (1 - \xi_n^+(\delta)) \langle J'((w, z)_\delta), (u_n, v_n) \rangle + \delta \xi_n^+(\delta) \langle J'((w, z)_\delta), \frac{J'(u_n, v_n)}{\|J'(u_n, v_n)\|} \rangle \\
& \leq o(\|(w, z)_\delta - (u_n, v_n)\|) + \frac{1}{n} \|(w, z)_\delta - (u_n, v_n)\|.
\end{aligned}$$

Divide the above inequality by  $\delta$  for  $\delta \neq 0$  and let  $\delta \rightarrow 0$ , then we get

$$\begin{aligned}
& -(\xi_n^+)'(0) \langle J'(u_n, v_n), (u_n, v_n) \rangle + \|J'(u_n, v_n)\| \\
& \leq (o(1) + \frac{1}{n})(1 + |(\xi_n^+)'(0)| \cdot \|(u_n, v_n)\|).
\end{aligned}$$

That is,

$$\|J'(u_n, v_n)\| \leq (o(1) + \frac{1}{n}) \cdot (1 + |(\xi_n^+)'(0)| \cdot \|(u_n, v_n)\|).$$

By (4.4), we only need to show  $|(\xi_n^+)'(0)|$  is uniformly bounded with respect to  $n$ . Noticing that

$$(\xi_n^+)'(0) = \langle (\xi^+)'(0, 0), \frac{J'(u_n, v_n)}{\|J'(u_n, v_n)\|} \rangle,$$

by (4.4) and the third assertion of Lemma 4.4, we can get that there exists  $C > 0$  such that

$$|(\xi_n^+)'(0)| \leq \frac{C}{|\|(u_n, v_n)\|^2 - 3A(u_n, v_n)|}.$$

Thus we only need to prove that  $|\|(u_n, v_n)\|^2 - 3A(u_n, v_n)|$  has a positive lower bound. Assume the contrary, then up to a subsequence,

$$\|(u_n, v_n)\|^2 - 3A(u_n, v_n) = o(1). \tag{4.7}$$

Since  $\{(u_n, v_n)\} \subset \mathcal{N}^+$ ,

$$\|(u_n, v_n)\|^2 - A(u_n, v_n) = B(u_n, v_n). \tag{4.8}$$

From (4.7) and (4.8) we have,

$$B(u_n, v_n) = \frac{2}{3} \|(u_n, v_n)\|^2 + o(1). \quad (4.9)$$

For fixed  $f(x), g(x)$ , since  $\max\{\|f\|_{\frac{4}{3}}, \|g\|_{\frac{4}{3}}\} < \Lambda$ , there must exist a small positive  $\tau$  such that

$$\max\{\|f\|_{\frac{4}{3}}, \|g\|_{\frac{4}{3}}\} < (1 - \tau)\Lambda. \quad (4.10)$$

By the proof of Lemma 2.1, one knows from the derivation of (2.3) that more accurate inequality will occur once (4.10) holds. That is, one has for  $\|(u, v)\| = 1$ ,

$$B(u, v) < \frac{2}{3}(1 - \tau) \sqrt{\frac{1}{3A(u, v)}}.$$

Thus by homogeneity,

$$B(u_n, v_n) < \frac{2}{3}(1 - \tau) \sqrt{\frac{\|(u_n, v_n)\|^2}{3A(u_n, v_n)}} \|(u_n, v_n)\|^2. \quad (4.11)$$

Dividing (4.11) by  $\|(u_n, v_n)\|^2$  and letting  $n \rightarrow \infty$ , we reach a contradiction from (4.4), (4.7) and (4.9), that

$$\frac{2}{3} \leq \frac{2}{3}(1 - \tau).$$

This completes the proof.  $\square$

On the other hand, we have

**Lemma 4.7** *Assume  $f(x), g(x) \in L^{\frac{4}{3}}(\Omega)$ , both nonzero, and  $\max\{\|f\|_{\frac{4}{3}}, \|g\|_{\frac{4}{3}}\} < \Lambda$ . Then there exists a sequence  $\{(u_n, v_n)\} \subset \mathcal{N}^-$ , such that when  $n \rightarrow \infty$ , it holds:*

1.  $J(u_n, v_n) \rightarrow \theta^-$ ;
2.  $J'(u_n, v_n) \rightarrow 0$ .

**Proof.** By Lemma 4.3,  $\mathcal{N}^-$  is closed in  $H$ . We use Ekeland's variational principle on  $\mathcal{N}^-$  to obtain a minimizing sequence  $\{(u_n, v_n)\} \subset \mathcal{N}^-$ , such that



1.  $J(u_n, v_n) < \inf_{(u,v) \in \mathcal{N}^-} J(u, v) + \frac{1}{n};$
2.  $J(w, z) \geq J(u_n, v_n) - \frac{1}{n} \|(w - u_n, z - v_n)\|$  holds for all  $(w, z) \in \mathcal{N}^-$ .

From Lemma 4.1 and the remark before Lemma 4.3 we obtain estimates similar to (4.4):

$$0 < \tilde{m} \leq \|(u_n, v_n)\| \leq \tilde{M},$$

where  $\tilde{m}$  and  $\tilde{M}$  are positive constants. By Lemma 4.5, the rest of the proof is similar to that of Lemma 4.6, we omit it.  $\square$

Since system (1.1) is subcritical, it is not difficult to obtain compactness conditions for the functional  $J$ . That is,

**Lemma 4.8** *Assume  $f(x), g(x) \in L^{\frac{4}{3}}(\Omega)$ , then  $J$  satisfies (PS) condition, i.e., for any  $c \in \mathbb{R}$ , any sequence  $\{u_n\} \subset H$  for which  $J(u_n) \rightarrow c$ ,  $J'(u_n) \rightarrow 0$  as  $n \rightarrow +\infty$  possesses a convergent subsequence.*

We are now able to give

**Proof of Theorem 1.1.** Firstly let us consider the minimization problem (3.2). By Lemma 4.6, there exists  $\{(u_n, v_n)\} \subset \mathcal{N}^+$  such that as  $n \rightarrow \infty$

$$J(u_n, v_n) \rightarrow \theta^+, \quad J'(u_n, v_n) \rightarrow 0.$$

Since  $J$  satisfies (PS) condition by Lemma 4.8, we find a  $(w_1, z_1) \in cl(\mathcal{N}^+) \subset \mathcal{N}^+ \cup \{(0, 0)\}$  such that  $J(w_1, z_1) = \theta^+$ ,  $J'(w_1, z_1) = 0$ . By Lemma 4.2,  $J(w_1, z_1) < 0$ . Thus  $(w_1, z_1) \neq (0, 0)$ , which implies  $(w_1, z_1) \in \mathcal{N}^+$ . We see  $(w_1, z_1)$  is a nontrivial weak solution of (1.1) by Lemma 3.2.

Furthermore, if  $f$  and  $g$  are both positive, we show the minimizer can be chosen to be a multiple of  $(|w_1|, |z_1|)$ . Indeed,  $\|(w_1, z_1)\| = \|(|w_1|, |z_1|)\|$ . Let

$$(|w_0|, |z_0|) := \frac{(|w_1|, |z_1|)}{\|(|w_1|, |z_1|)\|}, \quad (w_0, z_0) := \frac{(w_1, z_1)}{\|(w_1, z_1)\|}.$$

Since  $(w_1, z_1) \in \mathcal{N}^+$ , from the proof of Lemma 2.1 we know

$$B\left(\frac{w_1}{\|(w_1, z_1)\|}, \frac{z_1}{\|(w_1, z_1)\|}\right) > 0,$$

thus  $B(|w_1|, |z_1|) \geq B(w_1, z_1) > 0$ , which yields  $B(|w_0|, |z_0|) > 0$ . By the proof of Lemma 2.1, there exists  $t_1 > 0$  such that  $t_1 \cdot (|w_1|, |z_1|) \in \mathcal{N}^+$ . Since  $t_1 \|(w_1, z_1)\| \cdot (|w_0|, |z_0|) = t_1 \|( |w_1|, |z_1| )\| \cdot (|w_0|, |z_0|) = t_1 \cdot (|w_1|, |z_1|) \in \mathcal{N}^+$ , we know  $t_1 \|(w_1, z_1)\|$  is the first stationary point of the fibering map in the direction  $(|w_0|, |z_0|)$ .

Moreover,  $(w_1, z_1) \in \mathcal{N}^+$  is equivalent to  $\|(w_1, z_1)\|(w_0, z_0) \in \mathcal{N}^+$ , so  $\|(w_1, z_1)\|$  is the first stationary point of the fibering map in the direction  $(w_0, z_0)$ . Since  $B(|w_0|, |z_0|) \geq B(w_0, z_0) > 0$  and  $A(|w_0|, |z_0|) = A(w_0, z_0)$ , we can compare the above two roots of the associated fibering map to infer  $t_1 \|(w_1, z_1)\| \geq \|(w_1, z_1)\|$ . That is

$$t_1 \geq 1. \quad (4.12)$$

Taking account of the graph of the fibering map in direction  $(|w_1|, |z_1|)$ , one has from (4.12) and the fact  $t_1 \cdot (|w_1|, |z_1|) \in \mathcal{N}^+$  that,

$$J(t_1|w_1|, t_1|z_1|) \leq J(|w_1|, |z_1|).$$

Thus

$$\theta^+ \leq J(t_1|w_1|, t_1|z_1|) \leq J(|w_1|, |z_1|) \leq J(w_1, z_1) = \theta^+,$$

from which we know  $(t_1|w_1|, t_1|z_1|)$  solves problem (3.2). By Lemma 3.2 we see  $(t_1|w_1|, t_1|z_1|)$  is a weak solution of system (1.1).

Now we consider the minimization problem (3.3). By Lemma 4.7, there exists  $\{(u_n, v_n)\} \subset \mathcal{N}^-$ , such that as  $n \rightarrow \infty$

$$J(u_n, v_n) \rightarrow \theta^-, \quad J'(u_n, v_n) \rightarrow 0.$$

Since  $J$  satisfies  $(PS)$  condition by Lemma 4.8, we find a  $\{(w_2, z_2)\} \in cl(\mathcal{N}^-) = \mathcal{N}^-$  such that  $J(w_2, z_2) = \theta^-$ ,  $J'(w_2, z_2) = 0$ , and  $(w_2, z_2)$  is a nontrivial weak solution of (1.1) by Lemma 3.2.

Furthermore, if  $f$  and  $g$  are both positive, we show the minimizer for (3.3) can be chosen to be a multiple of  $(|w_2|, |z_2|)$ . By the proof of Lemma 2.1, there exists  $t_2 > 0$  such that  $t_2 \cdot (|w_2|, |z_2|) \in \mathcal{N}^-$ . Moreover, one calculates that the two fibering maps in direction  $(w_2, z_2)$  and  $(|w_2|, |z_2|)$  has the same turning point denoted  $t_0 = \sqrt{\frac{1}{3A(w_2, z_2)}} = \sqrt{\frac{1}{3A(|w_2|, |z_2|)}}$ . Thus  $t_2 > t_0$ , and by investigating the graph of the fibering map in direction  $(w_2, z_2)$  one gets

$$J(t_2 w_2, t_2 z_2) \leq J(w_2, z_2).$$

Now we have

$$\theta^- \leq J(t_2|w_2|, t_2|z_2|) \leq J(t_2w_2, t_2z_2) \leq J(w_2, z_2) = \theta^-,$$

so  $(t_2|w_2|, t_2|z_2|)$  solves (3.3). By Lemma 3.2 we see  $(t_2|w_2|, t_2|z_2|)$  is a weak solution of system (1.1).

We finish the proof by showing  $\theta_+ < \theta_-$ . In fact, if  $(w_2, z_2)$  is the minimizer of (3.3) satisfying  $B(w_2, z_2) \leq 0$ , then the associated fibering map has only one stationary point, which implies  $\theta_- > 0$  by the proof of Lemma 2.1. So  $\theta_+ < \theta_-$  by Lemma 4.2. On the other hand, if  $(w_2, z_2)$  satisfies  $B(w_2, z_2) > 0$ , then the associated fibering map  $\phi(t)$  has two stationary points:  $t_1, t_2(=1)$ . Thus from the graph of this fibering map, we get immediately that  $\theta_+ \leq \phi(t_1) < \phi(t_2) = \theta_-$ .  $\square$

**Remark** For the definition  $A(u, v)$  (2.1), we have by Hölder inequality

$$\int_{\Omega} u^2 v^2 \leq \left( \int_{\Omega} u^4 \right)^{\frac{1}{2}} \left( \int_{\Omega} v^4 \right)^{\frac{1}{2}} = \|u\|_4^2 \cdot \|v\|_4^2,$$

then

$$A(u, v) \geq \mu_1 \|u\|_4^4 + \mu_2 \|v\|_4^4 - 2|\beta| \|u\|_4^2 \|v\|_4^2 \geq 2(\sqrt{\mu_1 \mu_2} - |\beta|) \|u\|_4^2 \cdot \|v\|_4^2,$$

and the equality is satisfied for the second inequality if and only if  $\mu_1 \|u\|_4^4 = \mu_2 \|v\|_4^4$ . Thus as  $\|(u, v)\| = 1, \beta > -\sqrt{\mu_1 \mu_2}$ , we have

$$A(u, v) > 0.$$

Therefore, as  $\beta > -\sqrt{\mu_1 \mu_2}$ , all the proofs are valid, Theorem 1.1 above is still true.  $\square$

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## References

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